Student's Name:____________________________

Student Number: ___________________________

Teacher's Name: ___________________________

TEMPUS CELERIUS RADIO FUGIT

ABBOTSLEIGH

2012
HIGHER SCHOOL CERTIFICATE
Assessment 4

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Make sure your HSC candidate Number is on the front cover of each booklet.
- Start a new booklet for Each Question
- Answer the Multiple Choice questions on the answer sheet provided.

Total marks – (70)

- Attempt Sections 1 and 2
- All questions are of equal value

Section 1 Pages 2 - 6

10 marks
- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section 2 Pages 7 - 13

60 marks
- Attempt Questions 11– 14
- Allow about 1 hr and 45 minutes for this section

Outcomes to be assessed:

<u>Mathematics</u>

Preliminary :

A student

P1 demonstrates confidence in using mathematics to obtain realistic solutions to problems
P2 provides reasoning to support conclusions which are appropriate to the context
P3 performs routine arithmetic and algebraic manipulation involving surds, simple rational expressions and trigonometric identities
P4 chooses and applies appropriate arithmetic, algebraic, graphical, trigonometric and geometric techniques
P5 understands the concept of a function and the relationship between a function and its graph
P6 relates the derivative of a function to the slope of its graph
P7 determines the derivative of a function through routine application of the rules of differentiation
P8 understands and uses the language and notation of calculus

HSC :

A student

H1 seeks to apply mathematical techniques to problems in a wide range of practical contexts
H2 constructs arguments to prove and justify results
H3 manipulates algebraic expressions involving logarithmic and exponential functions
H4 expresses practical problems in mathematical terms based on simple given models
H5 applies appropriate techniques from the study of calculus, geometry, probability, trigonometry and series to solve problems
H6 uses the derivative to determine the features of the graph of a function
H7 uses the features of a graph to deduce information about the derivative
H8 uses techniques of integration to calculate areas and volumes
H9 communicates using mathematical language, notation, diagrams and graphs

<u>Mathematics Extension 1</u>

Preliminary:

A student

PE1 appreciates the role of mathematics in the solution of practical problems
PE2 uses multi-step deductive reasoning in a variety of contexts
PE3 solves problems involving permutations and combinations, inequalities, polynomials, circle geometry and parametric representations
PE4 uses the parametric representation together with differentiation to identify geometric properties of parabolas
PE5 determines derivatives which require the application of more than one rule of differentiation
PE6 makes comprehensive use of mathematical language, diagrams and notation for communicating in a wide variety of situations

HSC :

A student

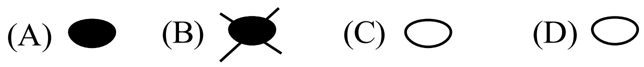
HE1 appreciates interrelationships between ideas drawn from different areas of mathematics
HE2 uses inductive reasoning in the construction of proofs
HE3 uses a variety of strategies to investigate mathematical models of situations involving projectiles.
HE4 uses the relationship between functions, inverse functions and their derivatives
HE6 determines integrals by reduction to a standard form through a given substitution
HE7 evaluates mathematical solutions to problems and communicates them in an appropriate form.

SECTION I

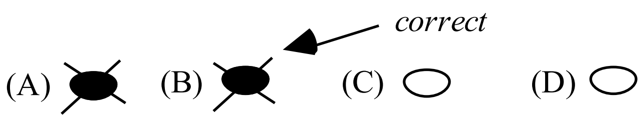
10 marks
Attempt Questions 1 – 10
Use the multiple-choice answer sheet
Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.



If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word *correct* and drawing an arrow as follows.



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1. The derivative of $e^{3x}\cos 3x$ is

(A) $-3e^{3x}\sin 3x$

(B) $e^{3x}(\cos 3x - 3\sin 3x)$

(C) $-9e^{3x}\sin 3x$

(D) $3e^{3x}(\cos 3x - \sin 3x)$

2. The solution to $\dfrac{x^2-9}{x} \geq 0$ is

(A) $-3 \leq x < 0;\ x \geq 3$

(B) $-3 \leq x \leq 3$

(C) $x \leq -3$ or $x \geq 3$

(D) $-3 \leq x \leq 3;\ x \neq 0$

3. The solution to $2\sin^2\theta - \sin\theta = 0$ for $0 \le \theta \le \pi$ is

(A) $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi$

(B) $\theta = 0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi$

(C) $\theta = \frac{\pi}{3}$ or $\frac{2\pi}{3}$

(D) $\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$

4. Given $f(x) = 2\sec x$ for $0 \le x \le \frac{\pi}{2}$, then $f^{-1}(x) =$

(A) $2\cos^{-1}\left(\frac{1}{x}\right)$

(B) $2\cos^{-1} x$

(C) $\cos^{-1}\left(\frac{2}{x}\right)$

(D) $\cos^{-1}\left(\frac{1}{2x}\right)$

5. The angle between the lines $2x + y = 4$ and $x + y = 2$, to the nearest degree, is

(A) 18°

(B) 25°

(C) 45°

(D) 72°

6. Evaluate $\lim_{x\to 0} \frac{\sin 3x}{5x}$

(A) 0

(B) Undefined

(C) $\frac{3}{5}$

(D) $\frac{5}{3}$

7. $P(x, y)$ divides AB externally in the ratio $2:3$ where $A(-1, 2)$ and $B\ (3, 5)$. Find P.

(A) $(11,\ 11)$

(B) $\left(\frac{3}{5}, \frac{16}{5}\right)$

(C) $(-3, -4)$

(D) $(-9, -4)$

8. $\int \sec 2x \tan 2x\, dx =$

(A) $-\frac{1}{2}\cos^3 2x + c$

(B) $\frac{1}{2}\tan^2 2x + c$

(C) $\frac{1}{2}\sec 2x + c$

(D) $\frac{1}{2}\sec^2 2x + c$

9. How many ways can 8 people, each with a different name, be arranged around a circular table if Jill must sit between Jack and Peter?

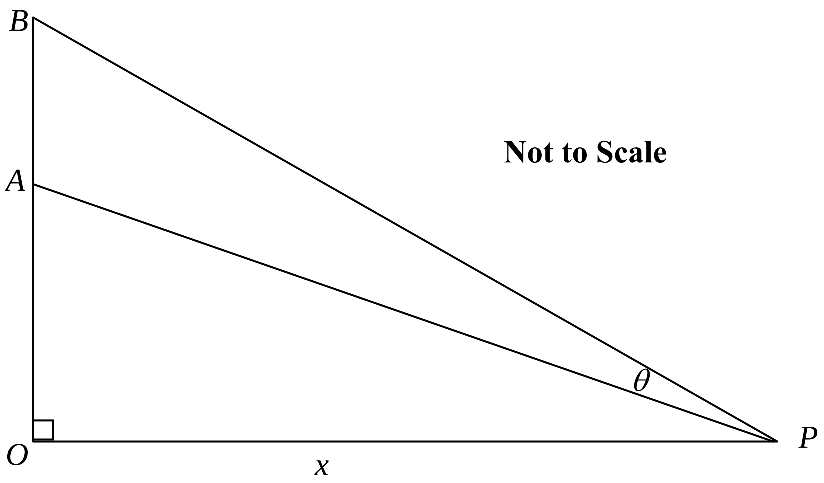
(A) 120

(B) 240

(C) 720

(D) 1440

10. In the diagram below, $\theta = \angle APB$ and $OA = 3\text{m}, AB = 1\text{m}$. Which equation is correct?



(A) $\theta = \tan^{-1}\left(\frac{1}{x}\right) - \tan^{-1}\left(\frac{3}{x}\right)$

(B) $\theta = \tan^{-1}\left(\frac{4}{x}\right) - \tan^{-1}\left(\frac{3}{x}\right)$

(C) $\theta = \tan^{-1}\left(\frac{1}{\sqrt{9 - x^2}}\right)$

(D) $\theta = \tan^{-1}(3 - x)$

End of Section I

SECTION 2

Total Marks – 60
Attempt Questions 11-14
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 11 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Express $\dfrac{{}^{10}C_k}{{}^{9}C_k}$ in simplest form. **2**

(b) A tennis team of 4 men and 4 women is to be selected from 6 men and 7 women.

(i) Find the number of ways in which this can be done. **1**

(ii) It was decided that two particular women must be selected together or not selected at all. How many teams could be selected in these circumstances? **2**

(c) Find $\int \cos^2 3x \, dx$ **2**

(d) Using the substitution $u = \log_e x$, evaluate $\displaystyle\int_1^e \frac{\log_e x}{3x}\, dx$ **3**

(e) Find the term independent of x in the expansion of $\left(x - \dfrac{1}{2x^3}\right)^{20}$ **3**

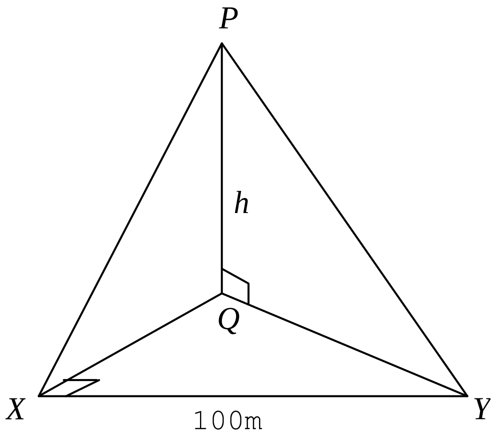
(f) Given the roots of $x^3 - x^2 + 4x - 2 = 0$ are α, β and γ, evaluate

(i) $\alpha\beta\gamma$ **1**

(ii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$. **1**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet. **Marks**



(a) The angle of elevation of a tower, PQ, from a point X due south of it is observed to be 70°. From another point Y, due east of X, the angle of elevation is 58°.

Given $XY = 100$ m, find the height h of the tower PQ to the nearest metre. **3**

(b) Consider the function $f(x) = x - 3 + \log_e x$.

(i) Show that there is a root between $x = 1$ and $x = 3$ **1**

(ii) Using the first approximation of the root $x_1 = 2$, use Newton's method to estimate the **second approximation** of this root.
Round your answer to 2 decimal places. **2**

(c) For the expansion of $(1+3x)^p(1-2x)^q$ find an expression for the coefficient of x^2. **2**

Question 12 continues on next page

(d) Consider the function $y = x\cos^{-1} x$

(i) Find $\dfrac{dy}{dx}$ **2**

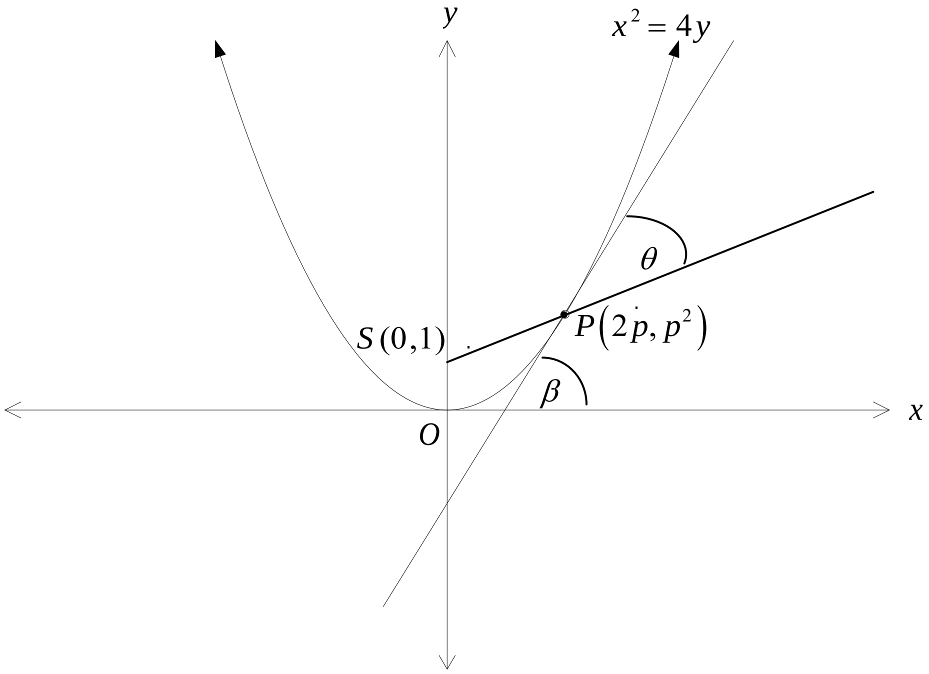
(ii) Hence, or otherwise, evaluate $\displaystyle\int_0^{\frac{\sqrt{3}}{2}} \cos^{-1}x \, dx$ **3**

(e) Using $\tan\dfrac{x}{2} = t$, or otherwise, prove that $\dfrac{2}{1+\cos x} = \sec^2\left(\dfrac{x}{2}\right)$ **2**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a)



Let $P(2p, p^2)$ be a point on the parabola $x^2 = 4y$, and let S be the focal point $(0, 1)$.
The tangent to the parabola makes an angle of β with the $x-$axis.
The angle between SP and the tangent is θ. Assume $p > 0$ as indicated.

(i) Show that $\tan\beta = p$ **1**

(ii) Show that the gradient of SP is $\frac{1}{2}\left(p - \frac{1}{p}\right)$ **1**

(iii) Show that $\tan\theta = \frac{1}{p}$ **2**

(iv) Show that the value of $\theta + \beta$ is $\frac{\pi}{2}$ **2**

(v) Find the coordinates of P when $\theta = \beta$ **1**

Question 13 continues on the next page

(b) Let $P(x) = x^3 + ax^2 + bx - 18$

Find the values of a and b if $(x+2)$ is a factor of $P(x)$, and -24 is the remainder when $P(x)$ is divided by $(x-1)$. **3**

(c) (i) Show that $\sqrt{3}\sin x + \cos x = 2\sin\left(x + \frac{\pi}{6}\right)$ **2**

(ii) Hence or otherwise sketch at least **one period** of the graph of $y = \sqrt{3}\sin x + \cos x$. Clearly show all $x-$intercepts and the amplitude. **2**
(Note: One period = one complete wave length)

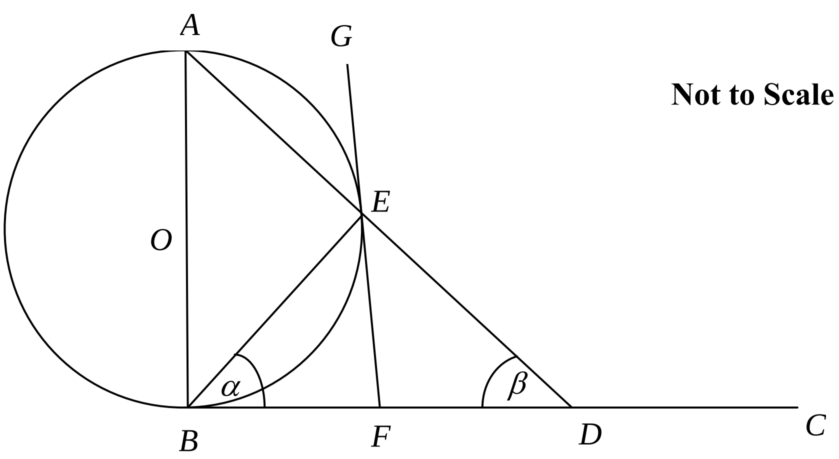
(iii) Using your graph, or otherwise, find the general solution of the equation $\sqrt{3}\sin x + \cos x = 0$ **1**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find the exact value of $\cos\left(\sin^{-1}\frac{3}{4}\right)$ **1**

(b) In the diagram below, AB is the diameter of the circle, centre O, and BC is tangential to the circle at B. The line AD intersects the circle at E and BC at D. The tangent to the circle at E intersects BC at F.
Let $\angle EBF = \alpha$ and $\angle EDF = \beta$.



(i) Copy the diagram into your writing booklet.

(ii) Prove that $\angle FED = \frac{\pi}{2} - \alpha$ **2**

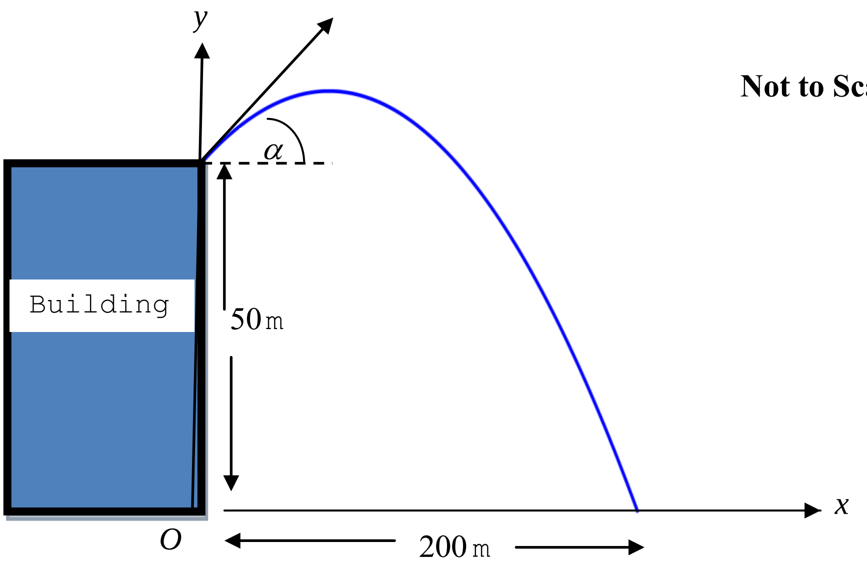
(iii) Prove that $BF = FD$ **2**

(c) Use mathematical induction to prove that, for all positive integers $n \geq 1$,

$$3.2^2 + 3^2.2^3 + 3^3.2^4 + \ldots\ldots + 3^n.2^{n+1} = \frac{12}{5}\left(6^n - 1\right)$$ **3**

Question 14 continues on next page

(d)



Not to Scale

The diagram shows the path of a projectile launched at an angle of elevation, α, with an initial speed of 40m/s from the top of a 50 metre high building and lands on the ground 200 metres from the base of the building. The acceleration due to gravity is assumed to be 10m/s^2.

(i) Given $\frac{d^2x}{dt^2} = 0$ and $\frac{d^2y}{dt^2} = -10$, show that $x = 40t\cos\alpha$ and $y = -5t^2 + 40t\sin\alpha + 50$, where x and y are the horizontal and vertical displacements of the projectile in metres from O at time t seconds after launching. **2**

(ii) If the projectile is launched at an angle of elevation of $45°$, find the maximum height it reaches above the ground. **2**

(iii) Find two possible values for α.
Give your answers to the nearest degree. **3**

END OF PAPER

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FORMULA SHEET ON BACK

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \; n \neq -1, \; x \neq 0, \; if \; n < 0$$

$$\int \frac{1}{x} dx = \ln x, \; x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \; a \neq 0$$

$$\int \sec^2 ax = \frac{1}{a} \tan ax, \; a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \; a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

YEAR 12

EXTENSION 1 2012

ASSESSMENT 4 SOLUTIONS

Section 1

1. C
2. A
3. B
4. C
5. A
6. D
7. D
8. C
9. B
10. B

Working

1. $\lim_{x \to 0} \frac{\sin 3x}{5x} = \lim_{x \to 0} \frac{\sin 3x}{3x} \cdot \frac{3}{5}$

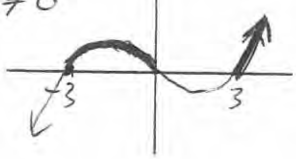
$= \frac{3}{5} \times 1$

$= \frac{3}{5}$

2. $x(x^2-9) \geq 0, \quad x \neq 0$

$x(x-3)(x+3) \geq 0$

$-3 \leq x < 0, \ x \geq 3$



3. $\sin\theta(2\sin\theta - 1) = 0$

$\sin\theta = 0$ or $\sin\theta = \frac{1}{2} \quad (0 \leq \theta \leq \pi)$

$\theta = 0, \pi \qquad \theta = \frac{\pi}{6}, \pi - \frac{\pi}{6}$

$\therefore \theta = 0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi$

4. $y = 2\sec x \quad \therefore f^{-1}: x = 2\sec y$

$x = \frac{2}{\cos y}$

$\therefore \cos y = \frac{2}{x}$

$y = \cos^{-1}\left(\frac{2}{x}\right)$

5. $m_1 = -2 \qquad m_2 = -1$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| \qquad \therefore \theta = \tan^{-1}\left(\frac{1}{3}\right)$

$= \left|\frac{-2+1}{1+2}\right| \qquad \doteq 18^\circ$

$= \frac{1}{3}$

6. $y = e^{3x}\cos 3x$

$y' = \cos 3x \cdot 3e^{3x} + e^{3x} \cdot -3\sin 3x$

$= 3e^{3x}(\cos 3x - \sin 3x)$

7. $A(-1, 2) \qquad B(3, 5)$

$-2 : 3$

$P = \left(\frac{3 \times -1 + 3 \times -2}{-2+3}, \frac{3 \times 2 + 5 \times -2}{-2+3}\right)$

$= (-9, -4)$

8. From table of standard integrals, 7th integral down page where $a = 2$

9. Place Jill, then 2 ways of placing Jack + Peter. Then 5! ways to place others

so $2 \times 5! = 240$

10. In $\triangle BOP$, $\tan\angle BPO = \frac{4}{x}$

$\therefore \angle BPO = \tan^{-1}\left(\frac{4}{x}\right)$

In $\triangle AOP$, $\tan\angle APO = \frac{3}{x}$

$\therefore \angle APO = \tan^{-1}\left(\frac{3}{x}\right)$

$\theta = \angle BPO - \angle APO$

$= \tan^{-1}\left(\frac{4}{x}\right) - \tan^{-1}\left(\frac{3}{x}\right)$

SECTION 2

11. (a) $\frac{{}^{10}C_k}{{}^{9}C_k} = \frac{10!}{k!(10-k)!} \div \frac{9!}{k!(9-k)!}$

$= \frac{10!}{k!(10-k)(9-k)!} \times \frac{k!(9-k)!}{9!}$

$= \frac{10}{10-k}$

(b) (i) $\frac{5!}{2!} = 60$

(ii) $\frac{3 \times 4 \times 3 \times 2 \times 1}{2!} = 36$

11.(c) $\int \cos^2 3x \, dx$

$= \frac{1}{2}\int (1+\cos 6x)\, dx$

$= \frac{1}{2}\left(x + \frac{1}{6}\sin 6x\right) + c$

(d) $\int_1^3 \frac{\log_e x}{3x} \, dx$

$u = \log_e x$

$du = \frac{1}{x} dx$

When $x=3$, $u = \ln 3$

When $x=1$, $u=0$

$= \int_0^{\ln 3} \frac{u}{3} \, du$

$= \left[\frac{u^2}{6}\right]_0^{\ln 3}$

$= \frac{(\ln 3)^2}{6}$

(e) General term ${}^{20}C_k \, x^{20-k}\left(-\frac{1}{2x^3}\right)^k$

$= {}^{20}C_k \, x^{20-k} \times x^{-3k} \times \left(-\frac{1}{2}\right)^k$

independent of x means x^0

$\therefore 20 - k - 3k = 0$

$k = 5$

$\therefore$ term is ${}^{20}C_5 \left(-\frac{1}{2}\right)^5 = -484.5$

(f) (i) $-\frac{d}{a} = \frac{--2}{1} = 2$

(ii) $\frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma} = \frac{\frac{4}{1}}{2}$

$= 2$

12(a) In $\triangle PQX$, $\tan 70° = \frac{h}{XQ}$

$\therefore XQ = \frac{h}{\tan 70°}$

In $\triangle PQY$, $\tan 58° = \frac{h}{QY}$

$\therefore YQ = \frac{h}{\tan 58°}$

$QY^2 = QX^2 + 100^2$ since $\angle QXY = 90°$

12(a) (Continued)

$\therefore 100^2 = \left(\frac{h}{\tan 58°}\right)^2 - \left(\frac{h}{\tan 70°}\right)^2$

$10000 = h^2\left(\frac{1}{\tan^2 58°} - \frac{1}{\tan^2 70°}\right)$

$h^2 \doteqdot \frac{10000}{0.257987375\ldots}$

$h \doteqdot \sqrt{38761.58661\ldots}$

$\doteqdot 196.879\ldots$

$\doteqdot 197$ m (to nearest metre)

(b) $f(x) = x - 3 + \log_e x$

(i) $f(1) = 1 - 3 + 0$

$= -2 < 0$

$f(3) = 3 - 3 + \log_e 3$

$= \log_e 3 > 0$

$\therefore$ root lies btwn 1 & 3 as function changes sign

(ii) $f'(x) = 1 + \frac{1}{x}$

Let $x_1 = 2$

$\therefore x_2 = 2 - \frac{f(2)}{f'(2)}$

$= 2 - \frac{(2 - 3 + \ln 2)}{1 + \frac{1}{2}}$

$\doteqdot 2 + 0.204\ldots$

$\doteqdot 2.20$ to 2 d.p.

(c) $(1+3x)(5-2x)^p$

$= (1+3x)\left({}^pC_0 5^p(-2x)^0 + {}^pC_1 5^{p-1}(-2x)^1 + {}^pC_2 5^{p-2}(-2x)^2 \ldots\right)$

$= (1+3x)\left({}^pC_0 5^p - 2\,{}^pC_1 5^{p-1}x + 4\,{}^pC_2 5^{p-2}x^2 + \ldots\right)$

$\therefore$ coeff of x^2 is $4\,{}^pC_2 5^{p-2} - 6\,{}^pC_1 5^p$

12(d) $y = x\cos^{-1}x$

(i) $\frac{dy}{dx} = \cos^{-1}x \cdot 1 + x \cdot -\frac{1}{\sqrt{1-x^2}}$

$= \cos^{-1}x - \frac{x}{\sqrt{1-x^2}}$

(ii) From (i):

$\int \cos^{-1}x - \int \frac{x}{\sqrt{1-x^2}} = x\cos^{-1}x$

$\therefore \int_0^{\frac{\sqrt3}{2}} \cos^{-1}x = \left[x\cos^{-1}x\right]_0^{\frac{\sqrt3}{2}} + \int_0^{\frac{\sqrt3}{2}} \frac{x}{\sqrt{1-x^2}}\,dx$

$= \left(\frac{\sqrt3}{2}\cos^{-1}\frac{\sqrt3}{2} - 0\right) + \int_0^{\frac{\sqrt3}{2}} x(1-x^2)^{-\frac12}\,dx$

$= \frac{\sqrt3}{2} \times \frac{\pi}{6} - \frac12 \int_0^{\sqrt3/2} -2x(1-x^2)^{-\frac12}\,dx$

$= \frac{\pi\sqrt3}{12} - \frac12 \left[\frac{(1-x^2)^{\frac12}}{\frac12}\right]_0^{\frac{\sqrt3}{2}}$

$= \frac{\pi\sqrt3}{12} - \left[\left(1-\frac34\right)^{\frac12} - (1)^{\frac12}\right]$

$= \frac{\pi\sqrt3}{12} - \frac12 + 1$

$= \frac{\pi\sqrt3 + 6}{12}$

e) Using $\tan\frac{x}{2} = t$,

LHS $= \frac{2}{1+\cos x}$

$= \frac{2}{1 + \frac{1-t^2}{1+t^2}}$

$= \frac{2(1+t^2)}{1+t^2+1-t^2}$

$= 1 + t^2$

$= 1 + \tan^2\left(\frac{x}{2}\right)$

$= \sec^2\left(\frac{x}{2}\right)$

= RHS

12(e) (alt. method)

let $x = 2\theta$

$\therefore$ LHS $= \frac{2}{1+\cos2\theta}$

$= \frac{2}{1+2\cos^2\theta-1}$

$= \frac{2}{2\cos^2\theta}$

$= \sec^2\theta$

$= \sec^2\frac{x}{2}$

= RHS

13.(a) $x^2 = 4y$ $P(2p, p^2)$
$S(0,1)$

(i) $y = \frac{x^2}{4}$

$\frac{dy}{dx} = \frac{x}{2}$

$\therefore$ grad. of tangent at $P = \frac{2p}{2}$
$= p$

Since $\beta = \angle$ btwn tangent + x axis
$\tan\beta$ = grad. of tangent at P
$= p$

(ii) $m_{SP} = \frac{p^2-1}{2p-0}$

$= \frac{p^2}{2p} - \frac{1}{2p}$

$= \frac12\left(p - \frac1p\right)$

(iii) θ = angle btwn SP & tang. at P

$\therefore m_1 = \frac12\left(p-\frac1p\right)$ $m_2 = p$

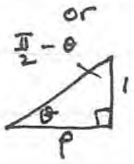
$\tan\theta = \left|\frac{m_1 - m_2}{1+m_1m_2}\right|$

$= \left|\frac{\frac12 p - \frac{1}{2p} - p}{1 + \frac{p^2}{2} - \frac12}\right|$

$= \left|\frac{-\frac{1}{2p}(1+p^2)}{\frac12(p^2+1)}\right| = \frac1p$

13(a) (Continued)

(iv) $\tan\theta = \frac{1}{p}$ $\tan\beta = p$



$\therefore \cot\theta = p = \tan\beta$

$\therefore \theta + \beta = \frac{\pi}{2}$ (complementary angles)

OR $\tan(\theta+\beta) = \dfrac{\tan\theta + \tan\beta}{1 - \tan\theta\tan\beta}$

$= \dfrac{\frac{1}{p} + p}{1 - 1}$

= undefined

$\therefore \theta + \beta = \frac{\pi}{2}$

(v) When $\theta = \beta$ $\frac{1}{p} = p$

$\therefore p = 1$, $P = (2, 1)$

b) $P(x) = x^3 + ax^2 + bx - 18$

$P(-2) = 0$ $\therefore -8 + 4a - 2b - 18 = 0$

$P(1) = -24$ $\therefore 1 + a + b - 18 = -24$

$4a - 2b = 26$ ①

$a + b = -7$ ②

$2a - b = 13$ ③

) + ③ $3a = 6$

$\underline{a = 2}$

$\underline{b = -9}$

) $\sqrt{3}\sin x + 1\cos x \equiv R\sin(x+\alpha)$

$R\sin x\cos\alpha + R\cos x\sin\alpha$

$\therefore \sqrt{3} = R\cos\alpha$

$1 = R\sin\alpha$

$R^2\cos^2\alpha + R^2\sin^2\alpha = 3 + 1 = 4$

$R^2(\sin^2\alpha + \cos^2\alpha) = 4$

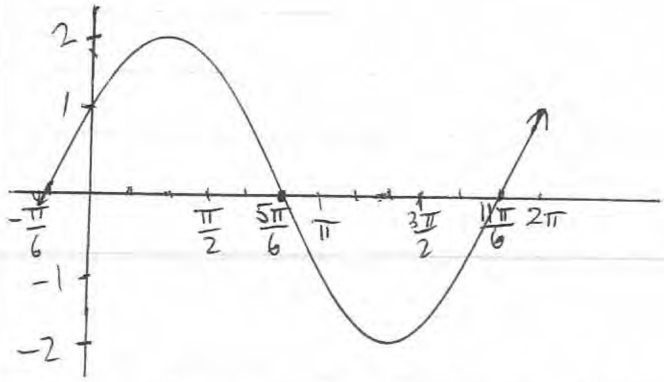
$R = 2$

$\tan\alpha = \dfrac{R\sin\alpha}{R\cos\alpha} = \dfrac{1}{\sqrt{3}}$ $\therefore \alpha = \frac{\pi}{6}$

$\sqrt{3}\sin x + \cos x \equiv 2\sin(x + \frac{\pi}{6})$

13(c) (Continued)

(ii) $y = 2\sin(x + \frac{\pi}{6})$

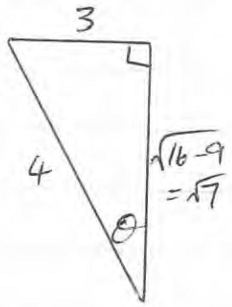


(iii) $x = k\pi - \frac{\pi}{6}$, k = any integer

OR $x = \frac{5\pi}{6} + k\pi$, k = any integer

14. (a) Let $\theta = \sin^{-1}\frac{3}{4}$

$\therefore \sin\theta = \frac{3}{4}$



$\therefore \cos\theta = \frac{\sqrt{7}}{4}$

(b)(ii) $FE = FB$ (tangents from external pt.)

$\therefore \angle BEF = \angle FEB = \alpha$ ($\angle$s opposite equal sides)

$\angle AEB = \frac{\pi}{2}$ ($\angle$ in semi circle)

$\therefore \angle BED = \frac{\pi}{2}$ (straight $\angle = 180°$)

$\therefore \angle FED = \angle BED - \angle BEF$

$= \frac{\pi}{2} - \alpha$

(iii) In $\triangle DEB$, $\beta = \pi - \frac{\pi}{2} - \alpha$ ($\angle$ su of $\angle$

$= \frac{\pi}{2} - \alpha$

$\therefore FD = FE$ (sides opposite equal $\angle$s)

But $FE = FB$ (tangents from external p

$\therefore FD = FB$

(c) $3\cdot2^2 + 3^2\cdot2^3 + \dots + 3^n\cdot2^{n+1} = \frac{12}{5}(6^n - 1$

Let $n = 1$ $\therefore$ LHS $= 3\cdot2^2$ RHS $= \frac{12}{5}(6$

$= 3\times4$ $= \frac{12}{5}\times5$

$= 12$ $= 12$

$\therefore$ true for $n = 1$

14(c) (Continued)

Assume true for $n=k$

$\therefore 3.2^2 + 3^2.2^3 + \cdots + 3^k.2^{k+1} = \frac{12}{5}(6^k - 1)$

Prove true for $n = k+1$

ie Prove $3.2^2 + \cdots + 3^k.2^{k+1} + 3^{k+1}.2^{k+2}$

$= \frac{12}{5}(6^{k+1} - 1)$

$LHS = \frac{12}{5}(6^k - 1) + 3^{k+1}.2^{k+2}$

$= \frac{12}{5}(6^k - 1) + 3^k.3.2^k.2^2$

$= \frac{12}{5}\left(6^k - 1 + \frac{12.6^k}{\frac{12}{5}}\right)$

$= \frac{12}{5}(6^k - 1 + 5.6^k)$

$= \frac{12}{5}(6.6^k - 1)$

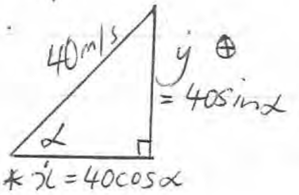
$= \frac{12}{5}(6^{k+1} - 1)$

$= RHS$

$\therefore$ If true for $n=k$, then also true for $n=k+1$. Is true for $n=1$ $\therefore$ also for $n=2$ $\therefore$ for $n=3$ etc for all positive integers $n \geq 1$.

(d) When $t=0$:

$x=0,\ y=50$



$\frac{d^2x}{dt^2} = 0$

$\frac{dx}{dt} = \int 0\,dt$

$= c$

$\frac{dx}{dt} = 40\cos\alpha$ from $*$

$\therefore x = \int 40\cos\alpha\,dt$

$x = 40t\cos\alpha + c$

When $t=0,\ x=0 \therefore c=0$

$\therefore x = 40t\cos\alpha$

$\frac{d^2y}{dt^2} = -10$

$\frac{dy}{dt} = -10t + c$

$\frac{dy}{dt} = -10t + 40\sin\alpha$ from ⊕

$y = \int(-10t + 40\sin\alpha)\,dt$

$y = -5t^2 + 40t\sin\alpha + c$

When $t=0,\ y=50 \therefore c=50$

$\therefore y = -5t^2 + 40t\sin\alpha + 50$

14(d) (Continued)

(ii) When the projectile lands

$y=0$ and $x=200$

$\therefore -5t^2 + 40t\sin\alpha + 50 = 0$ (1)

and $40t\cos\alpha = 200$ (2)

From (2) $t = \frac{5}{\cos\alpha}$

Sub. in (1):

$-5\left(\frac{25}{\cos^2\alpha}\right) + 40\sin\alpha\left(\frac{5}{\cos\alpha}\right) + 50 = 0$

$-125\sec^2\alpha + 200\tan\alpha + 50 = 0$

$-125(1+\tan^2\alpha) + 200\tan\alpha + 50 = 0$

$125\tan^2\alpha - 200\tan\alpha + 75 = 0$

$5\tan^2\alpha - 8\tan\alpha + 3 = 0$

$(5\tan\alpha - 3)(\tan\alpha - 1) = 0$

$\therefore \tan\alpha = \frac{3}{5}$ or $\tan\alpha = 1$

$\alpha \doteqdot 31^\circ$ or $\alpha = 45^\circ$

(iii) Let $\alpha = 45^\circ$

max y value when $\dot{y} = \frac{dy}{dt} = 0$

$\therefore -10t + 40\sin 45^\circ = 0$

$\frac{40}{\sqrt{2}} = 10t$

$t = \frac{4}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= 2\sqrt{2}$ seconds

$\therefore$ max. $y = -5(2\sqrt{2})^2 + 40(2\sqrt{2}).\frac{1}{\sqrt{2}} +$

$= -40 + 80 + 50$

$= 90$ metres.